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Diffusion in Hamiltonian systems driven by harmonic noise

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Abstract. We study the dynamics of randomly perturbed integrable Hamiltonian systems. In the limit of small perturbations, we show that the distribution function of the action variable satisfies a Fokker–Planck equation whose diffusion coefficient depends on the correlation function of the stochastic process. By using an harmonic noise we show the effect of resonances between the spectral density of the noise and the proper frequencies of the system. We explicitly consider the dynamics of a pendulum whose potential is stochastically perturbed; this model is relevant for the study of the synchrotron motion in accelerator physics. The numerical results for the distribution function of the energy are in very good agreement with the solutions of the Fokker–Planck equation.

1. Introduction

The effect of random perturbations on the solutions of a dynamical system is very important for the applications in [1–5]. Indeed any model which simulates the physical reality has to include a small noisy perturbation to take into account the effect of chaotic hidden degrees of freedom.

In this paper we study the dynamics of a one degree of freedom (1-d) Hamiltonian system which is stochastically perturbed. The main idea is to use an averaging theorem [6, 7] in the limit of small perturbation in order to prove that the distribution function of an invariant for the unperturbed system satisfies a Fokker–Planck (FP) equation. This result can be applied to a wide class of stochastic processes and the diffusion coefficient depends on the spectral density of the process itself. We explicitly consider the diffusion of the energy driven by a stochastic rotator [8] whose correlation function decays $\propto \mathrm{e}^{-\gamma t}\cos(\omega_0 t)$ and recall the correlation function of the harmonic noise which has previously been studied in the literature [9, 10]. The stochastic rotator has been chosen for two reasons: on one hand it allows us to point out the effect of the resonances between the unperturbed frequencies of motion and the noise spectrum. On the other hand we show numerical evidence that the correlation function of a chaotic variable which is defined on a compact domain, can be interpolated by an exponentially decaying oscillating function; as a consequence we conjecture that the diffusion of an integral of motion owing to a small coupling with a chaotic system could be simulated by using a stochastic rotator.

In this paper we explicitly compute the diffusion coefficient for the energy of a pendulum Hamiltonian system perturbed by a stochastic potential. This simple model is physically

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relevant in the study of the longitudinal (synchrotron) motion in a circular particle accelerator when a noise is inserted into radio-frequencies (RF) cavities [11–13]. We show that if the frequency ω_0 of the stochastic rotator satisfies a resonance condition with the frequency of an unperturbed orbit then even for a small noise amplitude the diffusion in the phase space can deteriorate the longitudinal stability of the bunched beam. If we remove the resonance condition keeping the noise amplitude constant, the diffusion velocity is reduced by a factor of $1/100$. The previous result is based on an averaging theorem which was recently generalized to 1-d Hamiltonian systems [14] and states that under certain conditions in the limit of vanishing noises ($\epsilon \to 0$), the unperturbed energy weakly converges towards a diffusion process whose distribution function satisfies a FP equation in the slow time $\tau = \epsilon^2 t$. The limit process is singular at the energy of the separatrix, so that we can solve the FP equation by inserting an artificial absorbing barrier near the separatrix. This is consistent with the application to accelerator physics, since the synchrotron motion for a bunched beam is confined in the region around the elliptic fixed point. Moreover, the instability in the synchrotron motion introduced by a stochastic perturbation could also be used as a mechanism for a slow extraction of the beam [15].

The numerical simulations which compare the distribution functions computed by integrating the equations of motion and the solution of the FP equation, show that we have a very good agreement even for finite values of the noise amplitude so that the results could be applied to physical experiments.

A simple, but not yet rigorous argument [16] shows that the relaxation time of the angle variable is much faster than the diffusion timescale $1/\epsilon^2$. This remark allows us to apply the averaging theorem in the case of a single realization of the noise, by using the angular initial conditions as probability space. The numerical simulations confirm the validity of this argument which is relevant in physical experiments where all of the particles feel the same realization of the noise.

The extension to the case of more degrees of freedom is not trivial owing to the presence of nonlinear resonances in any open set of the phase space. Different approaches to the analysis of stochastically perturbed Hamiltonian systems are discussed in [17, 18], where averaging principles are applied to the FP equation in an extended phase space or to the Liouville equation. At the diffusion timescale the distribution function of an unperturbed first integral of motion satisfies the same FP equation as in our approach.

2. Formulation of the model and theoretical results

Since our model refers explicitly to the synchrotron motion of a charged particle in a RF system, we briefly introduce the physical problem (for more details see [12]). A RF cavity is a device which produces an oscillating electrical field $\mathcal{E}(s, r, t)$ parallel to the reference orbit in a circular accelerator. When a particle with electrical charge e crosses a RF cavity its energy changes according to

$$\Delta E = \mathrm{e} \int \mathcal{E}(s, r, t)\, \mathrm{d}s \tag{1}$$

where s is the arc length of the reference orbit and r is the distance from the s-axis. The electrical field is given by $\hat{\mathcal{E}}(s, r) \sin(\omega_{RF} t)$ where ω_{RF} is the frequency of the cavity where the function $\hat{\mathcal{E}}(s, r)$ is even with respect to the centre of the cavity. Let $s = 0$ be the centre of the cavity and ϕ be the phase of the electric field when the particle is at $s = 0$, then we find the relation $\omega_{RF} t = \phi + \omega_{RF} s/v$ where v is the velocity of the particle and equation

(1) reduces to

$$\Delta E = e\sin\phi \int \mathcal{E}(s,r)\cos\left(\frac{\omega_{RF}\, s}{v}\right)\,\mathrm{d}s = eV(r)\sin\phi. \tag{2}$$

Since $r \ll 1$ it is possible to approximate $V(r) = V_0/N$ where N is the number of the cavities and the longitudinal motion is decoupled from the transverse motion. A particle which follows the reference orbit has the same phase ϕ_0 at each cavity along the magnetic lattice, as a consequence there is an integer ratio h between the ω_{RF} frequency and the revolution frequency ω_0. In contrast the phase of a particle outside the reference orbit changes between two nearby cavities according to

$$\Delta\phi_{n+1} - \Delta\phi_n = \frac{2\pi h}{N}\eta\frac{E - E_0}{E} \tag{3}$$

where $\Delta\phi = \phi - \phi_0$ denotes the difference with the phase of the reference particle and E_0 is the energy associated to the reference orbit. According to equation (2) the differences between the change of energy ΔE_0 of the reference particle and the change of energy ΔE of a generic particle can be written

$$\Delta E - \Delta E_0 = \frac{eV_0}{N}(\sin\phi - \sin\phi_0). \tag{4}$$

Finally we assume that the differences $\Delta\phi$ and ΔE are smooth functions so that $\Delta\phi = \dot{\phi}T/N$ and $\Delta E = \dot{E}T/N$ where T is the revolution period. By substituting these expressions into equations (3) and (4) we obtain the system

$$\begin{aligned} \dot{\phi} &= \frac{h\omega_0\eta}{E}(E - E_0) \\ \dot{E} - \dot{E}_0 &= -\frac{eV_0\omega_0}{2\pi}(\sin\phi - \sin\phi_0). \end{aligned} \tag{5}$$

It is possible to give a canonical form to system (5) by introducing the canonical variables $q = \phi$ and $p = (E - E_0)/h\omega_0$.

When the acceleration of the beam is completed ($\dot{E}_0 = 0$) the phase ϕ_0 of the reference particle is set to π and system (5) is a pendulum-like system. In this case the synchrotron motion is used to maintain the beam confined into different bunches. The presence of small noises in the RF cavities may introduce instabilities in the longitudinal beam confinement which reduce the luminosity of the machine.

In order to study the effect of a harmonic noise on the pendulum system we consider the dynamics in the extended phase space (q, p, x, y)

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -(1 + \epsilon x)\sin q \\ \dot{x} &= \dot{\omega}(t)y \\ \dot{y} &= -\dot{\omega}(t)x \end{aligned} \tag{6}$$

where the frequency $\omega(t)$ is defined

$$\omega(t) = \omega_0 t + \sqrt{2\gamma}\,w(t) \tag{7}$$

and $w(t)$ denotes the Wiener process. Equations (6) are integrated according to the Stratonovich interpretation in order to preserve the Hamiltonian character and the solution $(x(t), y(t))$ for the stochastic rotator is explicitly written

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos\omega(t) & \sin\omega(t) \\ -\sin\omega(t) & \cos\omega(t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \tag{8}$$

Equations (6) take into account the effect of a 'phase noise', whose amplitude is reduced to zero when we approach the fixed points $q = 0, \pi$; $p = 0$ of the unperturbed system.

The correlation function for the process (8) can be computed according to [19]

$$\begin{aligned}\langle x(t)x(t+s) + y(t)y(t+s)\rangle &= \langle (x^2(t) + y^2(t))\rangle \langle \cos(\omega(t+s) - \omega(t))\rangle \\ &= \langle (x^2(t) + y^2(t))\rangle \cos(\omega_0 s) \mathrm{e}^{-\gamma \langle w(T+s) - w(t)\rangle} = \langle (x^2(t) + y^2(t))\rangle \cos(\omega_0 s) \mathrm{e}^{-\gamma s}\end{aligned} \tag{9}$$

for $s > 0$ where $\langle\,\rangle$ denotes the expectation value with respect to the probability measure of $w(t)$ and the initial conditions $(x(0), y(0))$. If $(x(0), y(0))$ are defined on the unit circle as $x = \cos 2\pi\alpha$ and $y = \sin 2\pi\alpha$ where α is a random variable uniformly distributed in the interval [0,1], then one can easily prove that $\langle x(t)x(t+s)\rangle = \langle y(t)y(t+s)\rangle$ and the mean values $\langle x(t)\rangle$ and $\langle y(t)\rangle$ are zero. Finally, the correlation function of the process $x(t)$ which perturbs the pendulum equations is given by

$$\langle x(t)x(t+s)\rangle = \tfrac{1}{2}\cos(\omega_0 s)\mathrm{e}^{-\gamma s}. \tag{10}$$

The noise $x(t)$ is stationary and for $\omega_0 = 0$ has the same correlation of the Ornstein–Uhlenbeck process. Choice (10) for the correlation function is relevant to the study of the relation between the spectral density of the noise and the diffusion in the phase space of the unperturbed system; moreover, we show a numerical observation that the correlation function for a chaotic variable which is defined in a compact set has a similar oscillating behaviour. We then conjecture that the diffusion in system (6) could simulate the diffusion owing to a small coupling of an integrable system with a chaotic degree of freedom. Of course this result cannot be proved except for uniformly chaotic systems which satisfy a mixing condition, but we expect that the conjecture could be statistically true if the effect of trapping near the residual regular regions in the phase space (sticking phenomenon) affects a small number of orbits [21, 22]. In such a case we could explain the appearance of an anomalous diffusion [23–25] in the phase space, where the diffusion coefficient is larger than the well known quasilinear value. The analysis of this conjecture will be the object of a future work. As a model for a chaotic system, we consider the standard map

$$\begin{aligned}\theta_{n+1} &= \theta_n + r_{n+1} \bmod 2\pi \\ r_{n+1} &= r_n + K\sin\theta_n\end{aligned} \tag{11}$$

for a value of the parameter $K = 3$: the phase space is shown in figure 1 (top). Then we consider an ensemble of orbits which is obtained by choosing the initial conditions uniformly spread in the angle variable at $r_0 = 3$; such orbits fill the chaotic region between the two islands shown in figure 1. Then we numerically compute the correlation of the increments $\Delta r_n = K\sin\theta_n$

$$c(r_0; k, N) = \frac{\langle \Delta r_N \Delta r_{k+N}\rangle}{\langle \Delta r_N \Delta r_N\rangle} \tag{12}$$

where the mean value is taken over all the initial conditions. The value $N = 500$ is chosen in order to distribute the initial conditions in the whole chaotic region; this avoids transient effects since the initial set $\{r = r_0, \theta \in [0, 2\pi]\}$ is not invariant. In such a case the correlation function is practically independent from N and r_0. In figure 1(*b*) we show the numerical result (squares) for an ensemble of 20 000 orbits together with a numerical interpolation (broken line) obtained by using a function $\cos\omega_0 k \exp(-\gamma k)$ with $\omega_0 = 1.57 \simeq \pi/2$ and $\gamma = 0.115$. The numerical interpolation is good even if it is possible to observe that for large k the decaying of the correlation seems to be slower than an exponential decaying: a polynomial decaying fits apparently better the queue of the

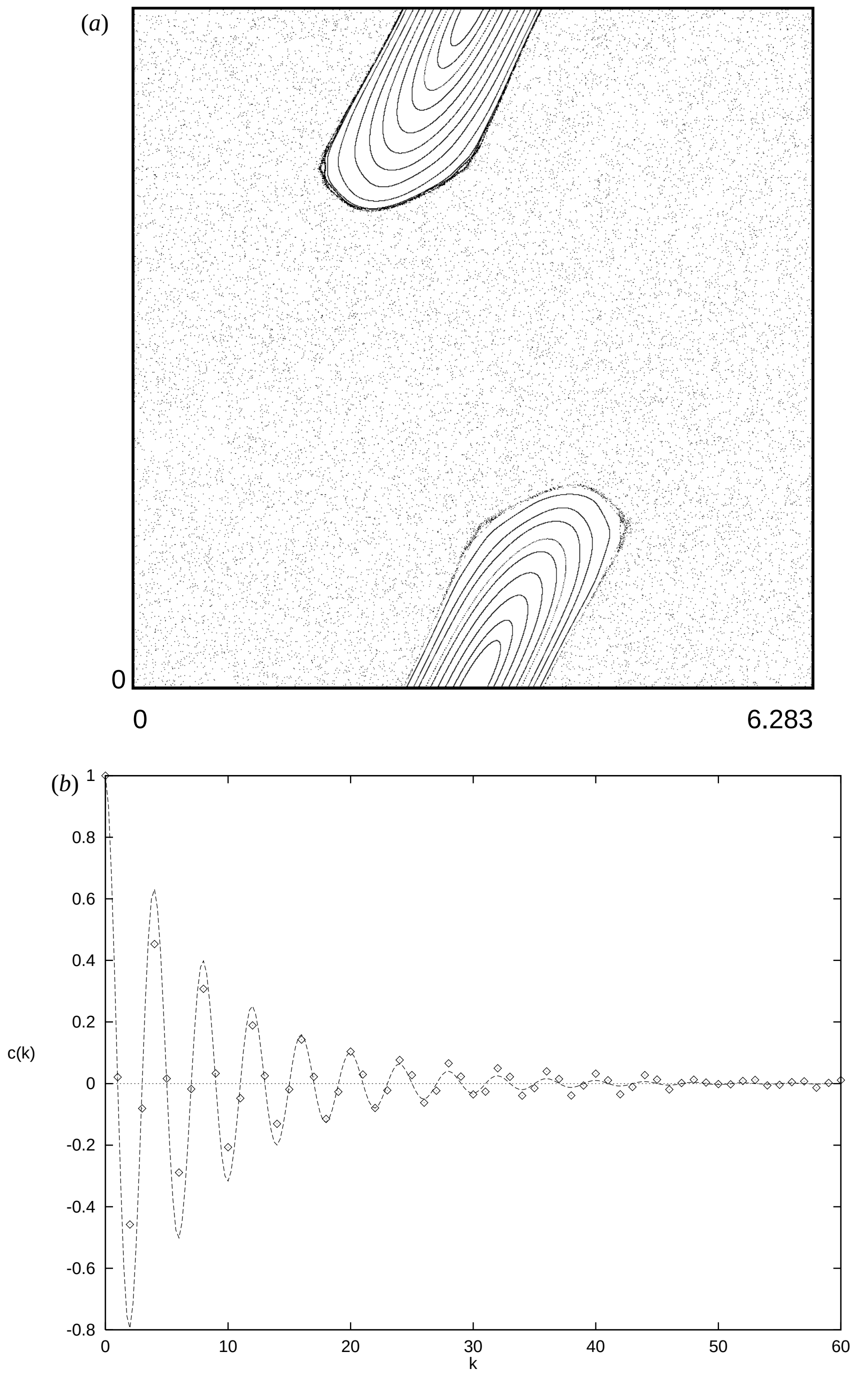

Figure 1. (*a*) Phase space of the standard map for a value $K = 3.0$ of the parameter. (*b*) Plot of the correlation function $c(k)$ (squares) computed using an ensemble of 20 000 orbits of the standard map with initial conditions uniformly distributed in the angle variable at $r_0 = 3$. The numerical results are interpolated by the function $\cos(\omega_0 t)e^{-\gamma t}$ (broken curve) $\omega_0 = 1.57$ and $\gamma = 0.115$.

correlation function. This is probably owing to the sticking phenomenon in the vicinity of the islands, which affects a certain number of orbits. Our numerical results are consistent with the computation of the correlation function of the standard map reported in the literature [26]. According to the conjecture, if we couple the chaotic motion of the phase variable θ with the dynamics of an integrable system, then we could observe a diffusion in the phase space of the integrable system which is similar to the diffusion driven by a stochastic rotator.

In order to study diffusion in the region bounded by the separatrix, we consider the distribution function for the unperturbed energy

$$E = \frac{p^2}{2} - \cos q. \tag{13}$$

We are in a situation to apply the following theorem [14].

Let us consider a stochastically perturbed Hamiltonian system

$$H(q, p, x(t)) = H_0(q, p) + \epsilon H_1(q, p, x(t)) \qquad (q, p) \in \mathbb{R}^2 \tag{14}$$

and let us introduce the action angle variables (θ, I), if the stochastic process $x(t)$ satisfies a ϕ-mixing condition (i.e. the events in the past and future can be considered independent up to an error of order $\phi(T)$ where T is the time gap) such that the following limit holds

$$\lim_{T\to\infty} T^6 \phi(T) = 0 \tag{15}$$

then in each compact set where the Hamiltonian field and its derivatives up to order 2 are bounded and the unperturbed frequency $\Omega(I) = \mathrm{d}H_0/\mathrm{d}I \neq 0$, the scaled process

$$\hat{I}(\tau) = I\left(\frac{\tau}{\epsilon^2}\right) \tag{16}$$

is weakly convergent to a Markov diffusion process in the slow time $\tau = \epsilon^2 t$ when the diffusion limit $\epsilon \to 0$, $t \to \infty$ is considered. The diffusion and the drift coefficients are defined according to

$$D(I) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} \mathrm{d}t \int_{t_0}^{t_0+T} \mathrm{d}s \left\langle \frac{\partial H_1}{\partial \theta}(\theta + \Omega t, I, x(t)) \frac{\partial H_1}{\partial \theta}(\theta + \Omega s, I, x(s)) \right\rangle \tag{17}$$

$$\mathrm{d}(I) = \frac{1}{2}\frac{\mathrm{d}D}{\mathrm{d}I}(I). \tag{18}$$

The existence of limit (17) is assumed.

We consider the case of a stationary random perturbation of the form $H_1(q, p, x(t)) = x(t)H_1(q, p)$ where $H_1(q, p)$ is an analytic function. If we introduce the Fourier expansion in the action-angle variables

$$H_1(\theta, I) = \sum_k h_k(I) \mathrm{e}^{\mathrm{i}k\theta} \tag{19}$$

the diffusion coefficient can be written in the form

$$D(I) = \sum_{k,k'} kk' h_{k'} h_k \mathrm{e}^{\mathrm{i}(k+k')\theta} \lim_{T\to\infty} \frac{2}{T} \int_0^T \mathrm{d}t \int_0^t \mathrm{d}s \phi(t-s) \mathrm{e}^{\mathrm{i}(k\Omega t + k'\Omega s)} \tag{20}$$

where $\phi(u)$ is the correlation function of the noise $x(t)$. Taking into account the fact that

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t \int_0^t \mathrm{d}s \phi(t-s) \mathrm{e}^{\mathrm{i}(k\Omega t + k'\Omega s)} &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t \mathrm{e}^{\mathrm{i}(k+k')\Omega t} \int_0^t \mathrm{d}u \phi(u) \mathrm{e}^{-\mathrm{i}k'\Omega u} \\ &= \delta_{k,-k'} \int_0^\infty \phi(u) \mathrm{e}^{-\mathrm{i}k\Omega u} \, \mathrm{d}u \end{aligned} \tag{21}$$

equation (20) reads

$$D(I) = \sum_k k^2 |h_k|^2 \tilde{\phi}(k\Omega) \tag{22}$$

where $\tilde{\phi}(\nu)$ is the spectral density of the noise

$$\tilde{\phi}(\nu) = \int_{-\infty}^{\infty} \phi(u) \cos \nu u \, \mathrm{d}u. \tag{23}$$

The drift and diffusion coefficients satisfy the conservative condition (18) so that the FP equation for the distribution function ρ_I in the action variable can be written

$$\frac{\partial}{\partial \tau} \rho_I(I, \tau) = \frac{1}{2} \frac{\partial}{\partial I} D(I) \frac{\partial}{\partial I} \rho_I(I, \tau). \tag{24}$$

In the previous calculations we explicitly use the nonresonant condition $\Omega(I) \neq 0$; this condition fails when we approach a separatrix $H_0(q, p) = E_*$ since the period of the orbits diverges as $\mathrm{O}(\log |E - E_*|)$. Moreover, the derivative $\partial H_1 / \partial \theta$ is singular at a separatrix because of the relation

$$\left. \frac{\partial H_1}{\partial \theta} \right|_{I=\text{constant}} = \frac{1}{\Omega(I)} \left(\frac{\partial H_1}{\partial q} \frac{\partial H_0}{\partial p} - \frac{\partial H_1}{\partial p} \frac{\partial H_0}{\partial q} \right) \tag{25}$$

so that the diffusion coefficient (22) diverges at the separatrix as $\mathrm{O}(1/\Omega^2)$. The FP equation (24) can be solved in any compact set contained in a region bounded by a separatrix with an absorbing boundary condition at the border.

The nonresonant condition ($\Omega(I) \neq 0$) means that the angle θ is a fast variable in the diffusion limit and we can average on that. Moreover, by using an heuristic argument [16], based on the assumption that $I(t)$ could be considered a diffusion process at a finite time, the integral

$$\int_0^t \Omega(I(s)) \, \mathrm{d}s \mod 2\pi \tag{26}$$

turns out to be a stochastic process on the circle which relaxes on a time of order $\epsilon^{-\frac{2}{3}}$ for an anisochronous system ($\mathrm{d}\Omega/\mathrm{d}I \neq 0$); as a consequence we can assume that the angle θ has a correlation time of the same order. According to this remark, even if we consider a single realization of the noise $x(t)$, in the diffusion limit two orbits of the system (14) with a different initial condition in the angle variable, behave as two different realizations of a stochastic process. This fact is relevant in the application to accelerator physics since the particles in the beam are perturbed by the same realization of noise, but have a different initial condition in the phase space.

In the diffusion limit the FP equation for the energy distribution function $\rho_E(E, \tau)$ is computed from the relation $\mathrm{d}E = \Omega(I)\mathrm{d}I$

$$\frac{\partial}{\partial \tau} \rho_E(E, \tau) = \frac{1}{2} \frac{\partial}{\partial E} \Omega(E) D(E) \frac{\partial}{\partial E} \rho_E(E, \tau) \Omega(E) \tag{27}$$

where we have defined $\Omega(E) = \Omega(I(E))$ and $D(E) = D(I(E))$. According to equation (27) the diffusion coefficient for the energy is $\Delta(E) = \Omega^2)E)D(E)$ and the drift coefficient is $\delta(E) = \Omega(E)d(E)$.

From equation (22) we see that the spectral density $\tilde{\phi}(\nu)$ contributes to the diffusion coefficient only with the amplitudes of the frequencies $\nu = k\Omega$ which enters in the Fourier expansion (19) of the perturbation; as a consequence the diffusion is enhanced when the spectral density of the noise is peaked at the frequencies $k\Omega$.

In the case of equations (6) $H_1(q, p, x(t)) = -x(t)\cos q$ and we can apply definition (22) for the diffusion coefficient where the spectral density of the noise is given by

$$\frac{1}{2}\int_{-\infty}^{\infty}\cos(\omega_0 s)\cos(\nu s)\mathrm{e}^{-\gamma s}\mathrm{d}s = \frac{1}{2}\left(\frac{\gamma}{\gamma^2+(\omega_0+\nu)^2}+\frac{\gamma}{\gamma^2+(\omega_0-\nu)^2}\right) \tag{28}$$

and it is peaked at $\nu = \pm\omega_0$.

3. Numerical results

In order to apply the results of the previous section to model (6), we have numerically evaluated the frequency $\Omega(E)$ and the Fourier expansion (19) with $H_1(\theta, I) = -\cos q(\theta, I)$, by using the discrete Fourier transform of the orbits $(q(t), p(t))$ $t \in \mathbb{N}$ of the unperturbed Hamiltonian $H = p^2/2 - \cos q$ [27]. The orbits are computed by means of a symplectic integrator of the order 4 with a step $\Delta t = 0.05$ up $t = 2000$ for a grid of 100 points in the energy interval $[-1, +1]$. Then we have evaluated numerically the frequency $\Omega(E)$ and the Fourier expansion of $H_1(\theta, I)$ up to the components $|k| \leqslant 6$ which provide a good precision in the values of the diffusion and drift coefficients. Of course the precision decreases as

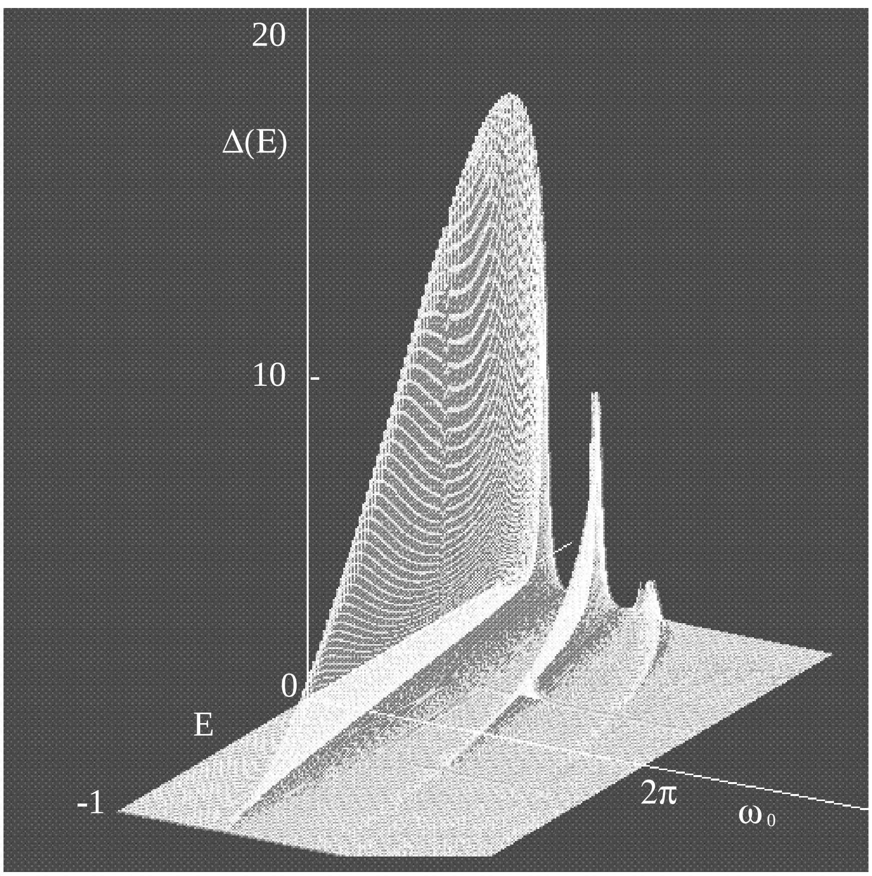

Figure 2. The figure shows the dependence of the energy diffusion coefficient from the parameter $\omega_0 \in [0, 2\pi]$ (x-axis) and the energy $E \in [-1, 1]$ (y-axis): the diffusion coefficient is numerically computed using the Fourier analysis of the unperturbed orbit.

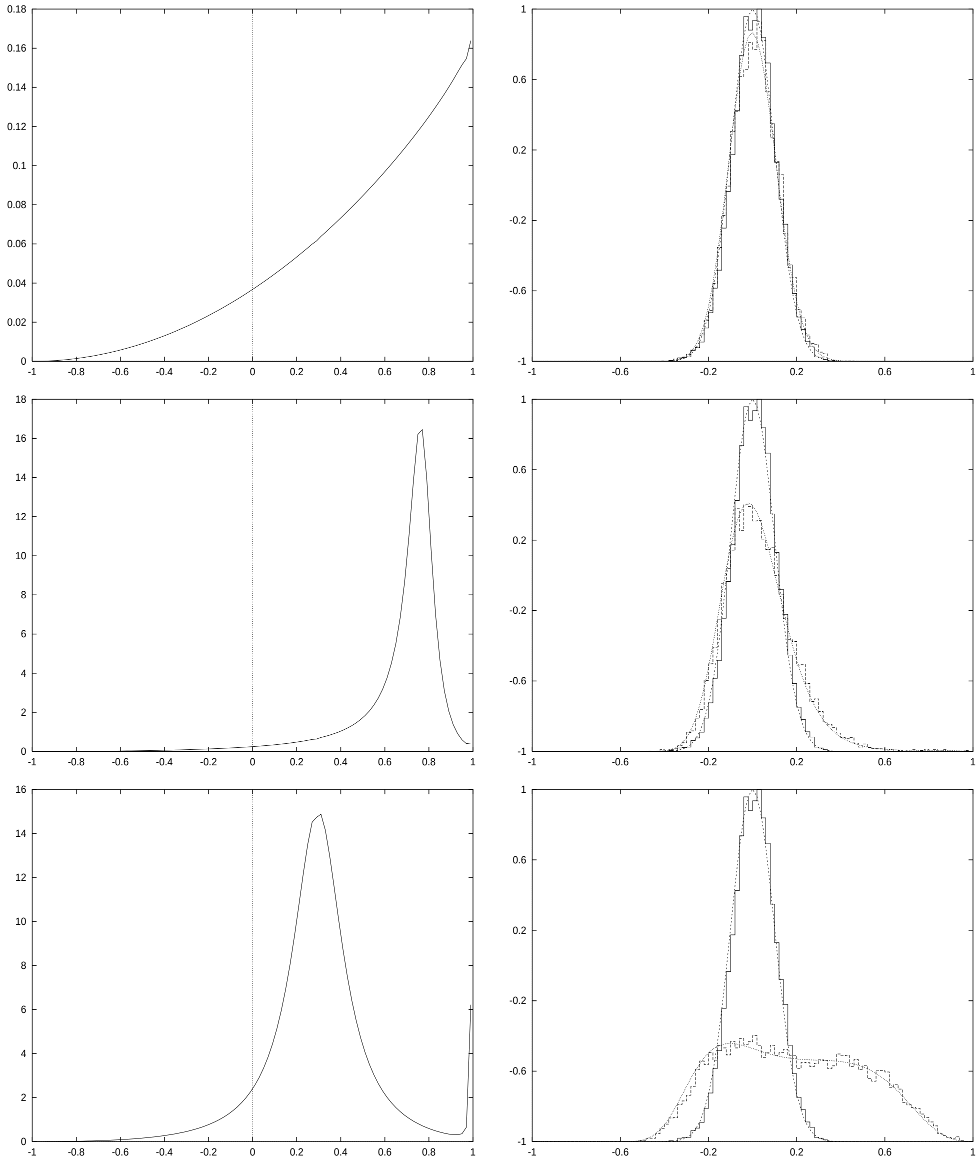

Figure 3. In the figures on the left-hand side we plot diffusion coefficients as a function of the energy $E \in [-1, +1]$ (note the different scales in the figures); the values of the parameters are $\epsilon = 0.02$, $\gamma = \frac{1}{15}$ and $\omega_0/2\pi = 0.05$, 0.2, 0.25 (respectively top, centre, bottom). In the figures on the right-hand side we compare the energy distribution functions computed by the numerical simulations (histogram) and the integration of the FP equation provided by the theory (full curve); we show the initial distribution and the distributions at $t_{\max} = 400$; a statistic of 10 000 realizations of the noise have been used in the simulations.

we approach the separatrix $E(q, p) = 1$. In figure 2 we plot the diffusion coefficient for the energy $\Delta(E, \omega_0) = \Omega(E)^2 D(E; \omega_0)$ as a function of $\omega_0 \in [0, 2\pi]$ and the energy $E \in [-1, 1]$.

We see clearly three peaks for the diffusion coefficient which correspond to the resonance conditions $\omega_0 = k\Omega(E)$ with $k = 2, 4, 6$ since only the even components appear in the Fourier expansion of $\cos q(\theta, I)$; according to equation (28) the amplitude of the peaks is $\propto 1/\gamma$ and the width is $\propto \sqrt{\gamma}$. Far from the resonant values for ω_0 the diffusion coefficient is $\propto \gamma$ so that the ratio with the diffusion coefficient near a resonant value is of order γ^2. In the limit $\omega_0 \to 0$ we recover the Ornstein–Ulhenbeck process which has been considered in other papers [4, 20] as a model to study the effect of a coloured noise.

The numerical method that we used to compute the diffusion coefficient $D(E)$ and the frequency $\Omega(E)$, is very general and can be applied to any Hamiltonian system.

In the numerical simulations we have considered various values of ω_0 in the correlation function (10) whereas the value of $\gamma = \frac{1}{15}$ was fixed. For each case we have numerically integrated the stochastic equations (6) with $\epsilon = 0.02$ by computing 10 000 realizations of the noise. Even if the theory applies in the limit $\epsilon \to 0$, the value 0.02 was chosen in order to limit the CPU time needed by the simulations. The initial condition for the distribution function $\rho_E(E, 0)$ is a Gaussian distribution centred at $E = 0$. In figure 3 (right) we compare the distribution functions ρ_E computed by the direct integration of equations (6) (histograms) and the same function (continuous curve) obtained from the integration of the FP equation (27) whose diffusion coefficient is plotted on the left: the pictures correspond to the values $\omega_0/2\pi = 0.05$, 0.2, 0.25 (respectively top, centre, bottom) and give the distribution ρ_E at $t = 0$ and $t_{\max} = 400$. The FP equation is integrated using a Cranck–Nicholson implicit algorithm with an absorbing boundary condition near $E = 1$ (separatrix curve) where the diffusion coefficient is singular and a reflecting boundary condition at the origin where the diffusion coefficient vanishes. The timestep and the energy step were chosen equal respectively to $t_{\max}\epsilon^2/100$ and $\frac{2}{100}$.

The averaging theorem does not apply in a domain which contains a separatrix curve, so that the absorbing boundary condition at the separatrix is necessary, however, for the applications to accelerator physics the particles that cross the separatrix are indeed lost, so that the absorbing barrier is physically justified. Equations (6) are integrated by using a symplectic integrator of order 4 with a timestep $\Delta t = 0.05$ and as soon as an orbit crosses the separatrix is considered lost.

In figure 4 we show the evolution of ρ_E for an ensemble of initial conditions in the phase space when we perturb the system (6) with a single realization of the noise $x(t)$. The left picture refers to the case $\omega_0 = 0.2$ and $\epsilon = 0.02$ and shows the distributions ρ_E at $t = 0$ and $t = 250$ computed by the simulations (histogram) and the FP equation: the initial condition are uniformly distributed in the angle variable with a Gaussian distribution in the energy. We observe that the agreement of the two curves is not satisfactory, but this result depends on our choice $\epsilon = 0.02$ which is too big to apply the theory so that the angle θ does not decorrelate sufficiently fast with respect to the diffusion velocity of the energy. Then we have reduced ϵ by a factor of 4 and we have multiplied the time for a factor of 16 in order to keep the diffusion time $\epsilon^2 t$ constant; the result for the distribution functions is plotted in figure 4(*b*), which shows a very good agreement between the simulations with a single noise realization and the solution of the FP equation.

4. Conclusions

We have studied the energy diffusion for stochastically perturbed Hamiltonian systems. According to an averaging theorem in the diffusion limit when the perturbation amplitude tends to zero, the energy weakly converges to a diffusion process in any compact subset of the region bounded by the separatrix. The drift and the diffusion coefficients are explicitly computed for a pendulum Hamiltonian in the case of a coloured noise and the effect of

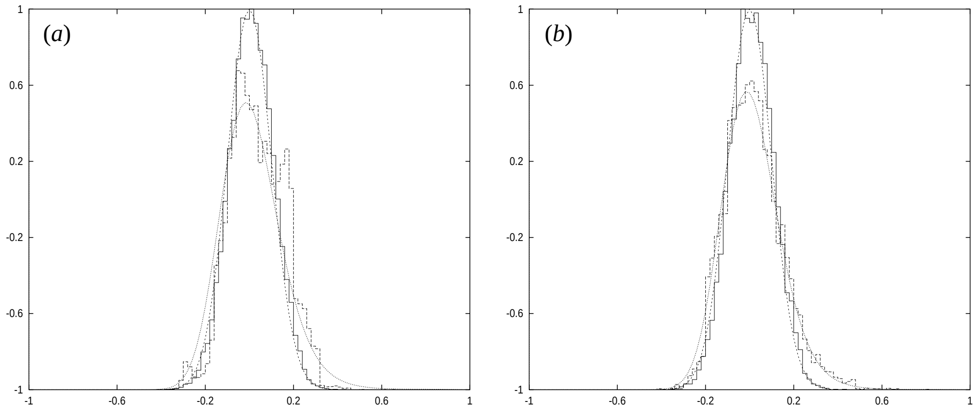

Figure 4. A comparison between the energy distribution functions in the case of a single realization of the noise and an ensemble of 10 000 initial conditions uniformly distributed in the angle variable; the histograms are the results of numerical simulations whereas the continuous curves are the solutions of the FP equation; the parameters used are $\gamma = \frac{1}{15}$, $\omega_0/2\pi = 0.2$ $\epsilon = 0.02, 0.005$ (respectively (a) and (b)); the distributions are plotted at $t = 0$, 250 (a) and $t = 0$, 4000 (b).

resonances between the proper frequencies of the pendulum and the frequencies contained in the noise spectrum has been pointed out. The comparison between the distribution function computed by a direct integration of the stochastic equations and the solution of the FP equation provided by the theory is also very good in the case of a finite amplitude of the perturbation. Moreover, the fast decorrelation in the angle variable which is expected in the anisochronous systems, allows us to describe the evolution of the energy distribution in the case of a single realization of the noise and an ensemble of initial conditions uniformly distributed in the angle variable.

The considered model is relevant in the applications to accelerator physics to study the effect of noise in the RF cavities on the stability of the synchrotron motion.

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